1 Measuring Figures and Objects

What You'll Learn

To measure the areas and perimeters of figures and the volumes of objects

And Why

To solve problems related to length of a fence, area of yard to be seeded, and volume of soil needed to fill a planter

Key Words

- leg
- hypotenuse
- inverse operation
- composite figure
- prism
- cylinder
- pyramid
- cone
- slant height
- sphere

Project Link

- Let's Play Mini-Golf!

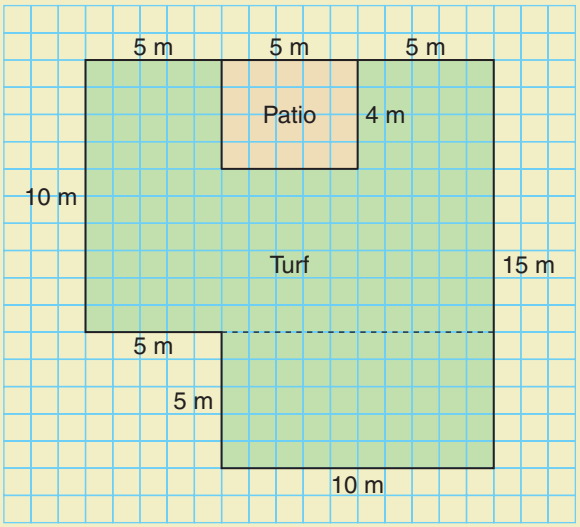
Reading and Writing in Math

Communicate Your Thinking

When you solve a problem, you can communicate your thinking in words, diagrams, or numbers.
Someone else should be able to understand your thinking and check your work.

Tina owns a landscape business.

Here is her sketch of a backyard and some calculations.
Check Tina's calculations.



Area of turf

$5 + 5 + 5 \times 10 + 5 \times 10 - 5 \times 4$
$= 10 + 50 + 50 - 20$
$= 110 - 20$
$= 90$

I think I need 90 m² of turf

Write a letter to Tina.
Point out the errors she made.
Describe the errors in detail.
Provide some advice so Tina does not make the same mistakes again.
Show Tina how to communicate her thinking better.

1.1 Measuring Perimeter and Area

Hyo Jin is redecorating her living room.
She measures the room.
Hyo Jin calculates the area of the walls to help determine how many litres of paint to buy.
She calculates the perimeter of the room to determine how many metres of wallpaper border to buy.

Investigate Perimeter and Area Problems

Work with a partner.
You will need scissors.
Your teacher will give you a larger copy of these clues.
Cut them apart.

➢ The clues belong in sets of 4.
Each set includes:
1 clue with a diagram,
1 clue describing a problem, and
2 clues with perimeter and area formulas.
Sort the clues into sets.

➢ Choose one set of clues.
Solve the problem.
Show your work.

	This perimeter is called circumference. $C = \pi d$ or $C = 2\pi r$	Perimeter is the sum of double the length and double the width. $P = 2l + 2w$	Area is one-half the area of the related parallelogram. $A = \frac{1}{2}bh$
	Area is the product of the length and width. $A = lw$		Perimeter is double the length of one side plus double the length of an adjacent side. $P = 2b + 2c$
Nicki is building a stage. The stage is a trapezoid. How much wood does she need for the floor of the stage?		Kevin is sewing lace trim on a circular tablecloth. How much trim does he need?	Perimeter is four times the length of one side. $P = 4s$
Trevor is putting a frieze around the rectangular ceiling of his studio. How much frieze does he need?	Perimeter is the sum of the lengths of all four sides. $P = a + b + c + d$	Area is the product of the length of one side and the height to the opposite parallel side. $A = bh$	
Amrit is painting handicapped parking spaces. Each space is a parallelogram. What is the area of each space?	Josie is laying sod on a triangular part of her property. How much sod does she need?	Keisha is tiling her square shower floor. How much tile does she need?	Area is the product of π and the square of the radius. $A = \pi r^2$
	Area is the square of the length of one side. $A = s^2$	Perimeter is the sum of the lengths of the three sides. $P = a + b + c$	Area is one-half the sum of the 2 parallel sides times the perpendicular distance between them. $A = \frac{1}{2}(a + b)h$

Reflect

➢ How did you solve the problem you selected?

➢ What is another way to solve the same problem?

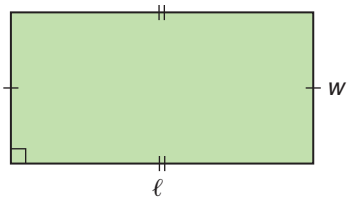
Connect the Ideas

The perimeter of a figure is the distance around it.

The area of a figure is the number of square units inside it.

One way to calculate perimeter or area is to substitute the appropriate measures into a formula.

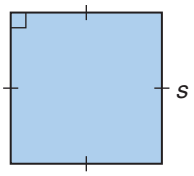
Rectangle



$P = 2\ell + 2w$

$A = \ell w$

Square

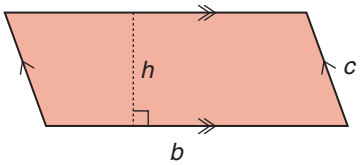


$P = 4s$

$A = s^2$

You can also find the perimeter of a figure whose sides are line segments by adding the lengths of its sides. You do not have to memorize the perimeter formulas.

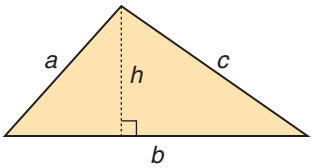
Parallelogram



$P = 2b + 2c$

$A = bh$

Triangle

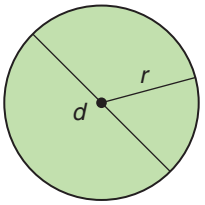


$P = a + b + c$

$A = \frac{1}{2}bh$

The circumference of the circle is also the perimeter of the circle.

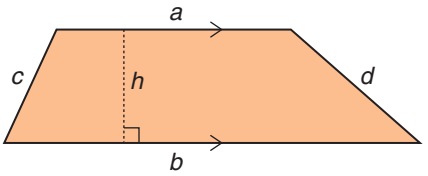
Circle



$C = \pi d$ or $C = 2\pi r$

$A = \pi r^2$

Trapezoid



$P = a + b + c + d$

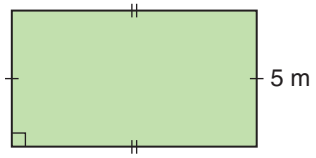
Area $= \frac{1}{2}$ (sum of parallel sides) × height

or, $A = \frac{1}{2}(a + b)h$

Practice

1. Determine the perimeter and area of each figure.

a)



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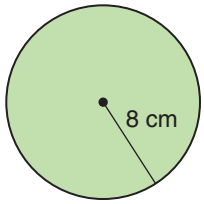

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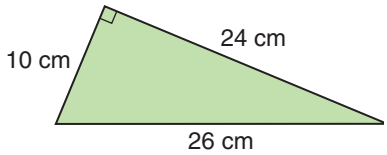
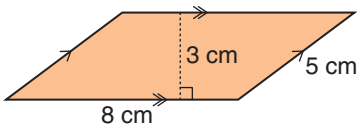


2. Determine the perimeter and area of each figure.

a)



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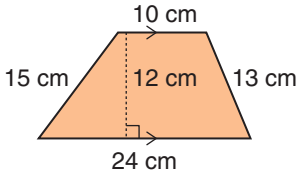

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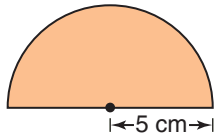
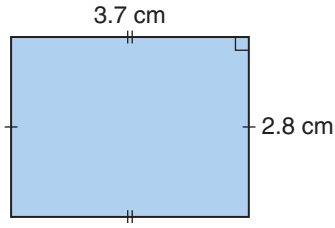


3. Determine the perimeter and area of each figure.

a)



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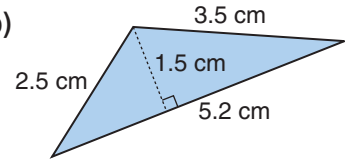

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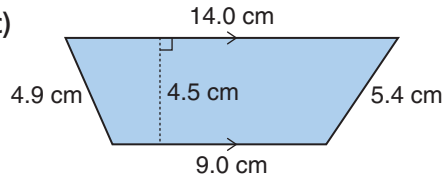
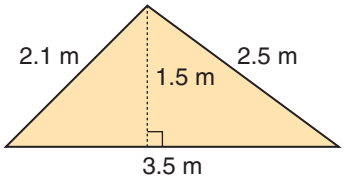
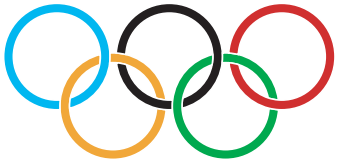


4. The sail on a yacht has the shape of a triangle.
What is the area of this sail?

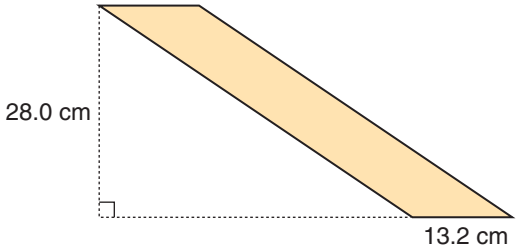
Remember to show your work.



5. Reanne is making the circles of the Olympic symbol from plastic tubing.
Each circle has radius 75 cm. How much tubing does she need?



6. When a paper towel tube is cut along its seam, it unwraps to form a parallelogram.
How much cardboard is used to make the tube?

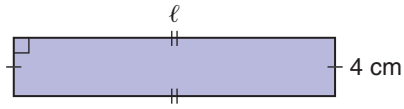


We can use a formula to determine the length or width when the perimeter is known.

Example

The perimeter of a rectangle is 56 cm. Its width is 4 cm. What is its length?

Solution



The perimeter P of a rectangle is:

$P = 2\ell + 2w$

Substitute: $P = 56$ and $w = 4$

$56 = 2\ell + 2(4)$

Solve for ℓ.

$56 = 2\ell + 8$ Think: What do we add to 8 to get 56?

We know that $56 = 48 + 8$

So, $2\ell = 48$ Think: What do we multiply 2 by to get 48?

We know that $2 \times 24 = 48$

So, $\ell = 24$

The rectangle is 24 cm long.

7. The area of a rectangle is 48 cm^2.

a) The width is 6 cm. What is its length?

b) The length is 12 cm. What is its width?

Having trouble?

Read the Example above.

8. Rosa has 24 m of fencing to make a square pen for her dog. How long is each side of the pen? Sketch the pen. Justify your answer.

9. **Assessment Focus** Serena has 3 m of garden edging. She wants to make a flowerbed that is an isosceles triangle.

a) Suppose each equal side is 90 cm long. How long is the third side?

b) Suppose the third side is 90 cm long. How long is each equal side?

Justify your answers.

Recall that 1 m = 100 cm.

10. **Take It Further** Luis makes a circle from a piece of wire 120 cm long.

a) What is the diameter of the circle?

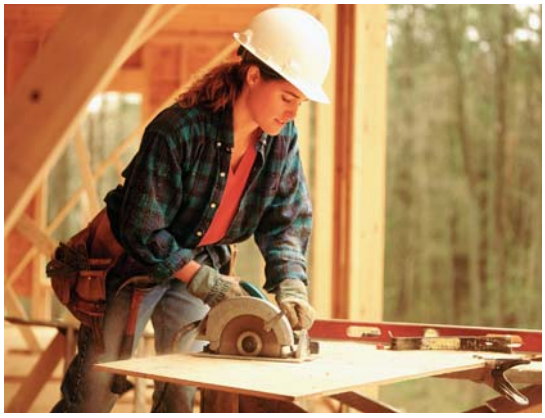
b) Will the wire fit around a circular tube with diameter 40 cm? Justify your answer.

In Your Own Words

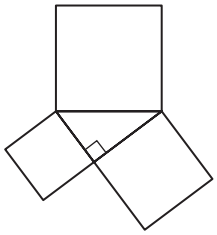
Choose one of the figures from this section. Explain how you found its perimeter or area.

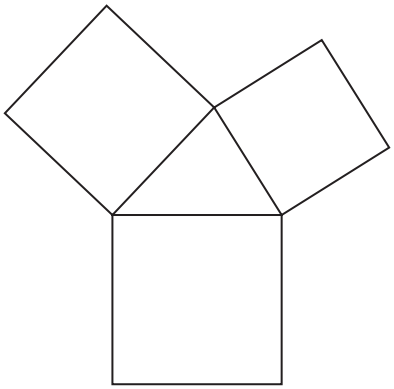
1.2 Measuring Right Triangles

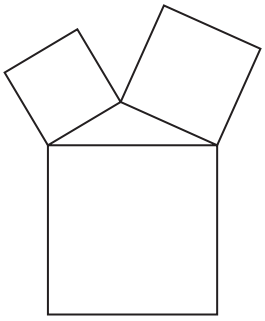
People in many occupations, such as carpenters, plumbers, and building contractors, use right triangles in their work.



Investigate Squares Drawn on the Sides of Triangles







You will need a ruler and protractor.
Your teacher will give you a larger copy of these triangles.

Each triangle has a square drawn on each side.
The side length of the square is the side length of the triangle.

- Begin with the right triangle.
  Calculate the area of each square.
  What do you notice about the 3 areas?
- Use a ruler and protractor.
  Draw a different right triangle and the squares on its sides.
  Are the areas of the squares related in the same way?
- Do you think the areas are related in the same way for each of these triangles?
  - an acute triangle
  - an obtuse triangle

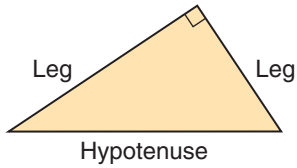
  To find out, repeat the activity for these triangles.

Reflect

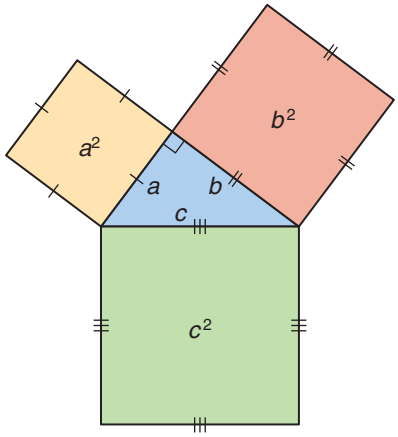
- Work with a classmate.
  Write a word equation that relates the areas of the squares on the sides of a right triangle.
  Include a sketch of a right triangle.
- Are the areas of the squares on the sides of an acute or obtuse triangle related in the same way?
  Justify your answer.

Connect the Ideas

In a right triangle, the shorter sides are the **legs**. These sides form the right angle. The longest side is opposite the right angle. It is called the **hypotenuse**.



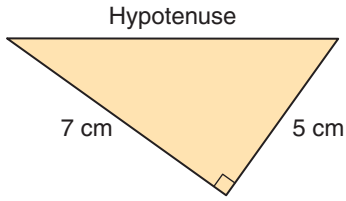
The sum of the areas of the squares on the legs is equal to the area of the square on the hypotenuse.



This relationship is written as:
$a^2 + b^2 = c^2$

This relationship is named the Pythagorean Theorem. The theorem is not true for acute or obtuse triangles. This relationship can be used to determine the length of the hypotenuse of a right triangle when you know the lengths of the legs.

To determine the hypotenuse in this right triangle, substitute for a and b in the formula $a^2 + b^2 = c^2$.
Substitute: $a = 7$ and $b = 5$



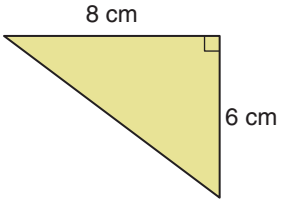
$$7^2 + 5^2 = c^2$$
$$(7 \times 7) + (5 \times 5) = c^2$$
$$49 + 25 = c^2$$
$$74 = c^2$$ Use the **inverse operation**.
$$c = \sqrt{74}$$ You know c^2. To find c, take the square root.
$$c \doteq 8.602$$ Use the $\sqrt{\ }$ key on a calculator.

The hypotenuse is about 8.6 cm, to 1 decimal place.
This result is reasonable. That is, the hypotenuse is always greater than the legs and 8.6 is greater than 5 and 7.

Practice

Where necessary, give the answers to 1 decimal place.

1. Determine each unknown length.

a)


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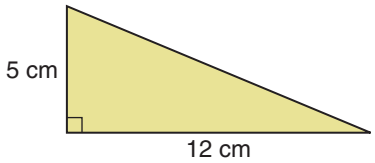

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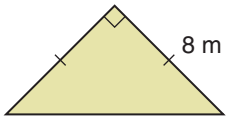
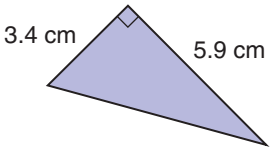


2. Determine each unknown length.

a)


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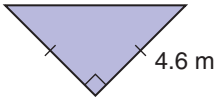

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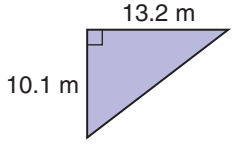
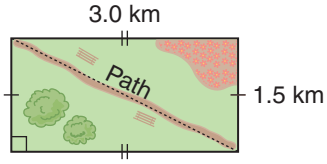


3. Ali walks along the path through the park.
How far does Ali walk?

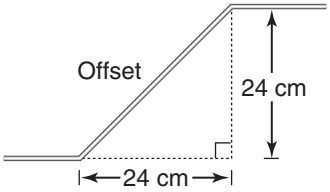


4. A right triangle has legs 9 cm and 12 cm.
Sketch the triangle. What is its perimeter?
Show your work.

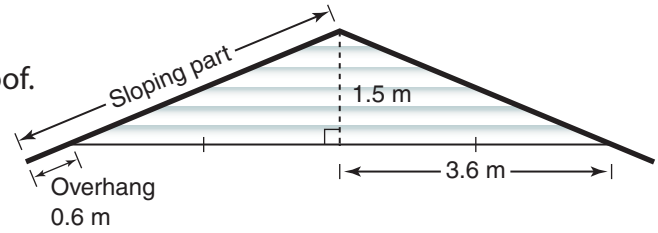
5. A plumber is installing pipe.
She has to offset the pipe around an obstacle.

a) What is the length of this offset section of pipe?
Justify your answer.

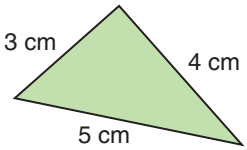
b) Is the result reasonable? Explain.



6. A contractor estimates how many sheets of plywood will be needed to build this roof.
His first step is to determine the length of the sloping part.
How long is it? Justify your answer.



7. Use the Pythagorean Theorem to find out if these are right triangles.
Justify your answers.

a)


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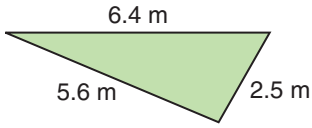

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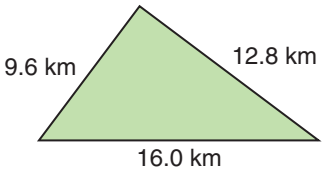


Remember: $a^2 + b^2 = c^2$ is only true for right triangles.
c is the longest side.

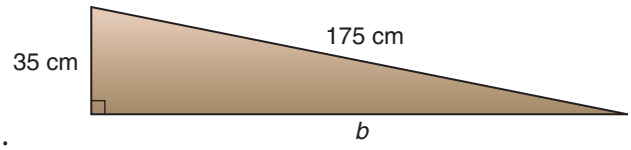
We can also use the Pythagorean Theorem to determine the length of a leg.

Example

Kim is building a ramp with a piece of wood 175 cm long.
The height of the ramp is 35 cm.
What is the horizontal length of the ramp?

Solution

Since the side view of the ramp is a right triangle, we use the Pythagorean Theorem.



Let the horizontal leg be b.

Use: $a^2 + b^2 = c^2$

Substitute: $a = 35$ and $c = 175$

$$35^2 + b^2 = 175^2$$
$$1225 + b^2 = 30\,625$$
$$1225 - 1225 + b^2 = 30\,625 - 1225$$
$$b^2 = 29\,400$$
$$b = \sqrt{29\,400}$$
$$b \doteq 171.46$$

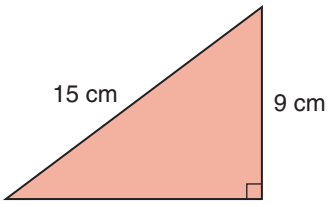
To isolate b^2, subtract 1225 from each side of the equation.

To calculate b, take the square root.

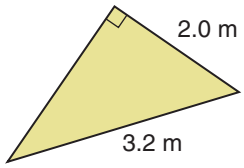
The horizontal length of the ramp is about 171 cm.

8. Determine each unknown length.

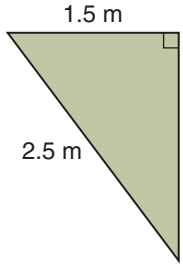
a)



b)



c)



9. **Assessment Focus** Suppose you know that the lengths of two sides of a right triangle are 3.5 cm and 4.5 cm. What is the length of the third side? Show two possible answers.

10. **Take It Further** A ladder is 4.9 m long. It leans against a wall with its foot 1.2 m from the base of the wall. The distance from the foot of a ladder to the wall should be about one-quarter of the distance the ladder reaches up the wall.
Is the ladder safely positioned? Justify your answer.

In Your Own Words

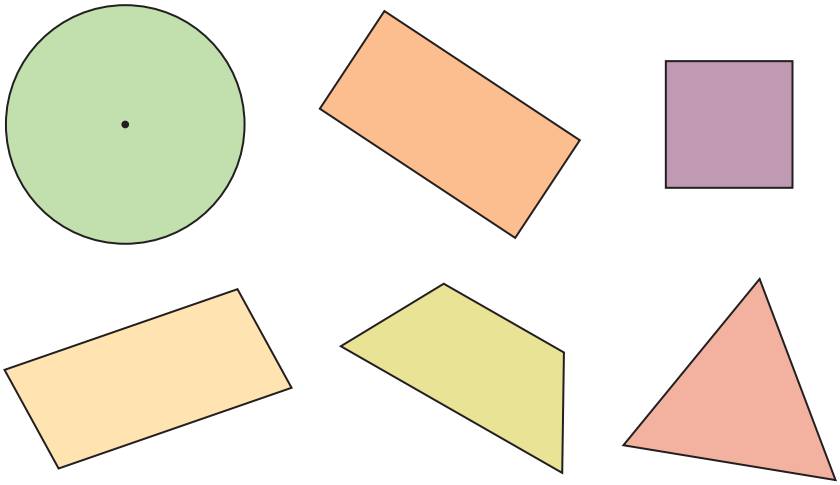
Describe how the areas of the squares drawn on the sides of a right triangle are related.
Why is it important to know the Pythagorean Theorem?

1.3 Area of a Composite Figure

An appraiser is determining the value of some vacant land.
As part of the process, he must calculate the area of the land.
Often parcels of land have irregular shapes.
How can their areas be calculated?

Investigate Drawing a Composite Figure and Determining Its Area

You may need dot paper or grid paper.
You know how to determine the area of each figure below.



- Combine any or all of these figures, or parts of them.
  Design an unusual garden.
- Determine the area of your garden.
  Write enough measures on your design so someone else could calculate its area.

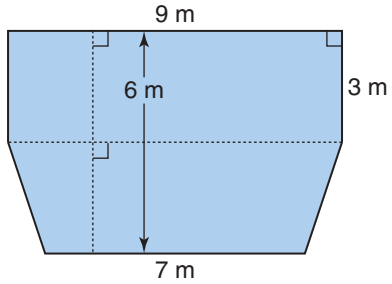
Reflect

- Trade designs with a classmate.
  Calculate the area of your classmate's garden.
- Compare answers. If both of you have different answers for the same garden, try to find out why.

Keep your designs for use in Section 1.4.

Connect the Ideas

A figure made up of other figures is called a **composite figure**. This composite figure is made up of a rectangle and a trapezoid.



Determine the area of each part

The rectangle has dimensions 9 m by 3 m.

Its area is: $A = \ell w$
$= 9 \times 3$
$= 27$

The trapezoid has parallel sides of 9 m and 7 m.

The height of the trapezoid is: $6\text{ m} - 3\text{ m} = 3\text{ m}$

Its area is: $A = \frac{1}{2}(a + b)h$
$= \frac{1}{2}(9 + 7) \times 3$
$= \frac{1}{2} \times 16 \times 3$
$= 8 \times 3$
$= 24$

Determine the total area

Total area = Rectangle area + Trapezoid area
$= 27 + 24$
$= 51$

The area of the composite figure is 51 m^2.

Is the answer reasonable?

From the diagram, the area of the trapezoid is a little less than the area of the rectangle.

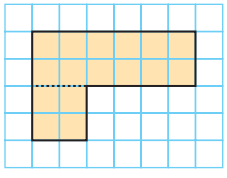
So, the total area should be less than twice the area of the rectangle. That is, less than $2 \times 27 = 54$

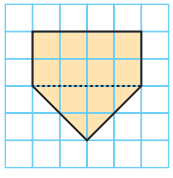
Since 51 is a little less than 54, the answer is reasonable.

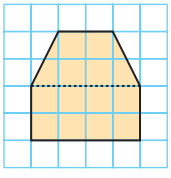
Practice

Assume the figures in questions 1 and 2 are drawn on 1-cm grid paper.

1. Determine the area of each composite figure.

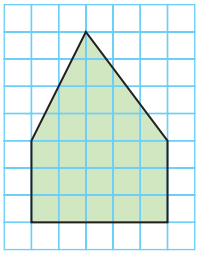
a)


b)


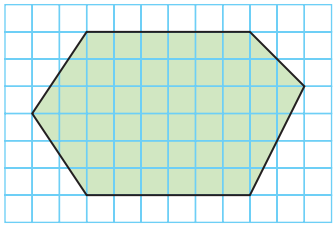
c)


2. Determine the area of each composite figure.

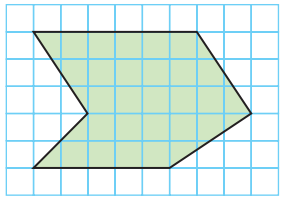
a)



b)

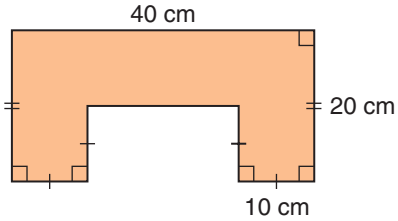


c)

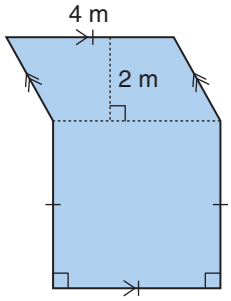


3. Determine the area of each composite figure.

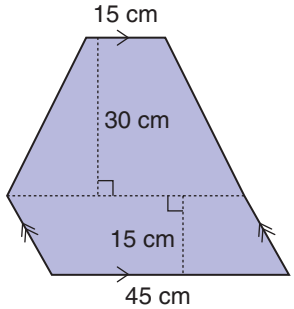
a)



b)



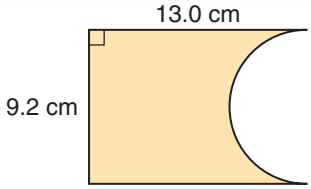
c)



Another way to determine the area of a composite figure is to draw a figure around the composite figure first.

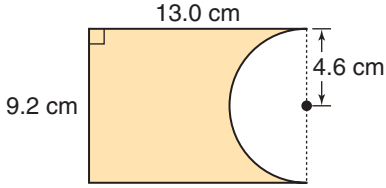
Example

Determine the area of this composite figure. The curve is a semicircle.



Solution

The composite figure is a rectangle that measures 9.2 cm by 13.0 cm, with a semicircle removed.

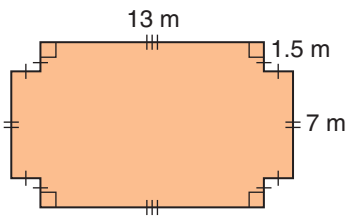


- The diameter of the semicircle is 9.2 cm.
  So, the radius of the semicircle is:
  $\frac{9.2\text{ cm}}{2} = 4.6\text{ cm}$
  The area of a circle is: $A = \pi r^2$
  So, the area of the semicircle is:
  $A = \frac{1}{2}\pi r^2$
  Substitute: $r = 4.6$
  $A = \frac{1}{2} \times \pi \times (4.6)^2 \doteq 33.24$
- The area of the rectangle is: $9.2 \times 13.0 = 119.6$
- Total area = Rectangle area − Semicircle area
  $\doteq 119.6 - 33.24$
  $= 86.36$

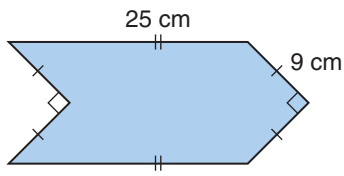
The area of the composite figure is about 86.4 cm^2.

4. Calculate the area of each figure.

a)

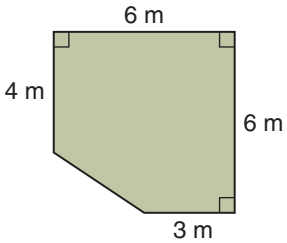


b)

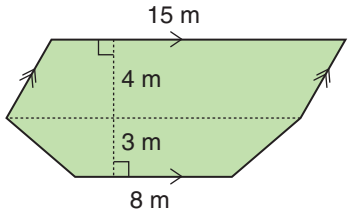


5. Kiren quotes for paving driveways based on the area to be paved. Determine the area of each driveway. All curves are semicircles. Is each result reasonable? Explain.

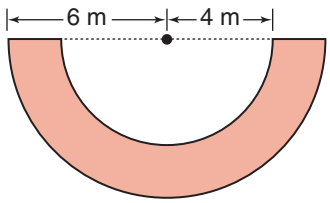
a)



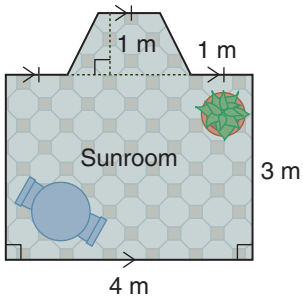
b)



c)



6. **Assessment Focus** The floor plan of a sunroom is shown.



a) What is the area of the floor of this sunroom?

b) There are 10 000 cm^2 in 1 m^2.
What is the area of the floor in square centimetres?

c) A square tile has side length 30 cm.
What is the area of 1 tile?

d) Estimate how many tiles are needed to cover the floor.
Justify your answer.

7. **Take It Further** One flowerbed has the shape of a trapezoid.
The parallel sides are 5 m and 7 m long.
The distance between the parallel sides is 4 m.
On the shorter parallel side, there is another flowerbed that has the shape of a rhombus, with side length 5 m and height 3 m.

a) Sketch the flowerbeds.

b) What is the area of the flowerbeds?

c) One bag of topsoil covers 0.25 m^2. How many bags are needed?

d) One bag of topsoil costs $2.29, including taxes.
How much will the topsoil in part c cost?

In Your Own Words

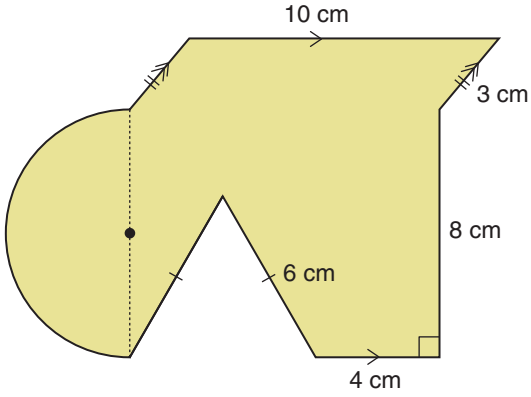
Sketch and label a composite figure.
Determine its area. Try to do this two different ways if possible.
Show your work.

1.4 Perimeter of a Composite Figure

Tamara works for a fencing company. She is preparing a price quote for a customer. Tamara needs to know the type of fencing being ordered and the perimeter of the area to be fenced.

Investigate Perimeter of a Composite Figure

Your teacher will give you a large copy of this design. The curve is a semicircle.

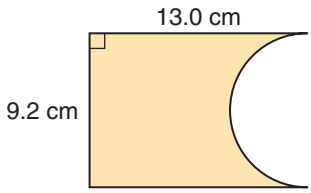


Or, you can use your garden design from Section 1.3. Calculate the perimeter of the design.

Reflect

- Trade solutions with a classmate. Check your classmate's solution.
- Compare answers. If you have different answers for the same garden, try to find out why.
- For which figures are the measurements you need for perimeter different from those you need for area? How did you find these measurements?

Connect the Ideas



Here is a composite figure from Section 1.3.
The perimeter of this figure is the sum of
3 sides of a rectangle and
one-half the circumference of a circle.

Determine the curved length

The diameter of the circle is 9.2 cm.
The circumference of a circle is: $C = \pi d$
So, the circumference is: $C = \pi \times 9.2$
One-half the circumference is: $\frac{1}{2} \times \pi \times 9.2 \doteq 14.45$

Determine the perimeter

The approximate perimeter of the composite figure is:
$13.0 + 9.2 + 13.0 + 14.45 = 49.65$
The perimeter is about 49.7 cm.

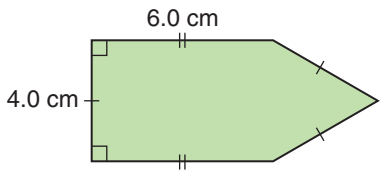
Check the result

From the diagram, the length of the semicircle
is greater than the width of the rectangle.
So, the perimeter of the figure should be greater than
the perimeter of the rectangle, which is approximately
$2(9) + 2(13) = 18 + 26 = 44$; the result is reasonable.

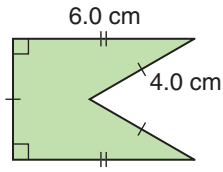
Practice

1. **a)** Determine the perimeter of each composite figure.

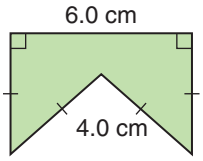
i)



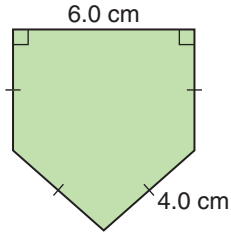
ii)



iii)

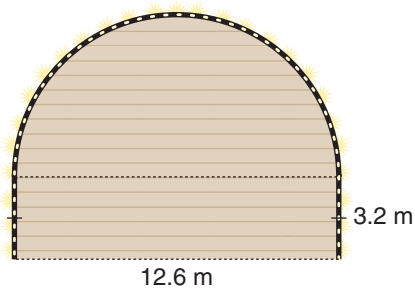


iv)



b) What do you notice about the perimeters in part a?
Do you think the same relationships are true
for the areas? How could you find out?

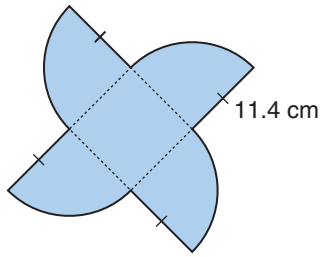
2. Sarah is lighting the theatre stage arch for the new play.
The arch is a semicircle on top of a rectangle.
How long is the string of lights? Justify your answer.



3. This design is 4 one-quarter circles on the sides of a square.

a) What is the perimeter of the design? The broken lines are not part of the perimeter.

b) Is your result reasonable? Explain.

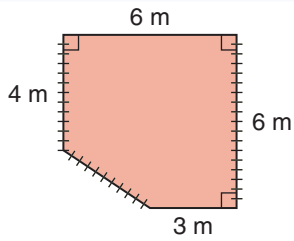


Sometimes you may need to use the Pythagorean Theorem to calculate a length before you can determine the perimeter.

Example

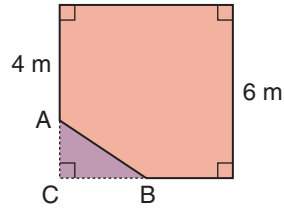
Here is a plan of a driveway from Section 1.3.

A fence is to be placed around the driveway on the sides indicated. How much fencing is needed?



Solution

We know the length of each part of the fence except for AB.
Draw right △ABC.
Then AC = 6 m − 4 m = 2 m
and BC = 6 m − 3 m = 3 m
To find the length of AB, use the Pythagorean Theorem in △ABC.



$a^2 + b^2 = c^2$

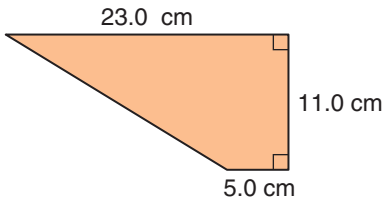
Substitute: $a = 3$ and $b = 2$

$$\begin{aligned} 3^2 + 2^2 &= c^2 \\ 9 + 4 &= c^2 \\ 13 &= c^2 \\ c &= \sqrt{13} \\ c &\doteq 3.6 \end{aligned}$$

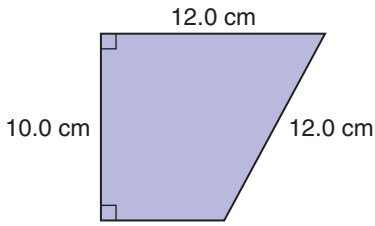
The total length of fencing is:
6 m + 3.6 m + 4 m = 13.6 m
About 14 m of fencing are needed.

4. Determine the perimeter of each figure. Show your work.

a)



b)



Need Help?
Read the Example on page 17.

5. **Assessment Focus** A circular fish pond is set in a rectangular patio.

a) Plastic edging is placed around the pond and the patio.

i) What length of edging is used?

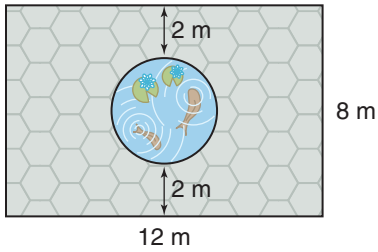
ii) The edging costs \$4.79/m.
What is the total cost of the edging?

b) The patio is paved with sandstone tiles.

i) What is the area that is paved?

ii) The sandstone costs \$45.00/m^2.
What is the total cost of the sandstone?

c) What assumptions did you make in parts a and b?



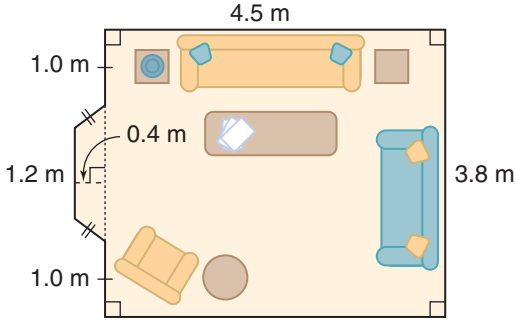
6. Sketch two different composite figures that have the same perimeter.
Calculate the perimeters, or explain how you know they are equal.
Calculate each area.

7. **Take It Further** Yazan is putting up a wallpaper border in his family room.
The border will run along the top of all the walls, including above any doors and windows.

a) What length of border does Yazan need?

b) The border comes in 4.57-m rolls.
How many rolls does Yazan need?

c) Each roll of border costs \$14.99.
How much will the border for this room cost?

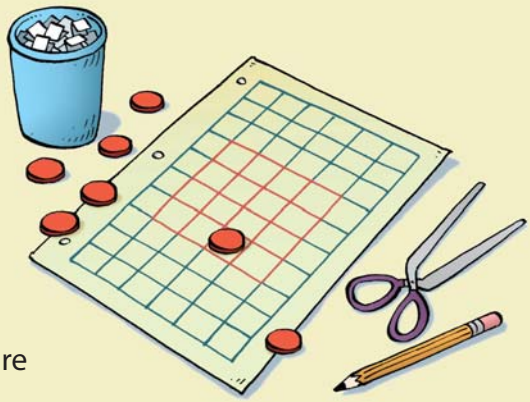


In Your Own Words

Sketch a composite figure.
Explain how you calculate its perimeter.
Show your work.

Measurement Bingo

GAME



Materials:

- counters as markers
- 2-cm grid paper
- scissors
- container

Play in a group of 4.

➢ Each player outlines a 4 by 4 square on 2-cm grid paper.

➢ Write these 16 answers anywhere on your 4 by 4 grid — one in each square:
circumference of a circle; area of a circle; area of a square; trapezoid; cm^3; m^2; 6; 9; 10; 11; 20; 24; 27; 30; 34; 45

➢ Your teacher will give your group a copy of 16 questions.

➢ Cut the questions apart.
Fold each question, and place it in a container.

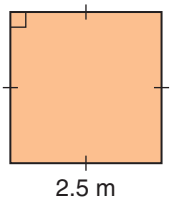
➢ Take turns to pick a question.
Everyone answers the question.
Cover the answer on your grid with a marker.

➢ The first person to cover a row or column wins.

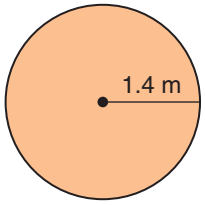
Mid-Chapter Review

1.1 **1.** For each figure:
- Determine its area.
- Determine its perimeter.

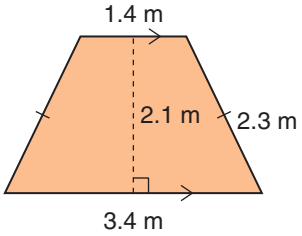
a)



b)



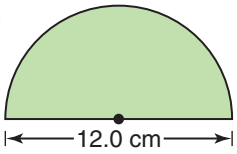
c)



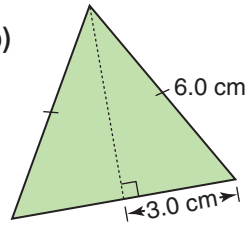
2. For each measurement in question 1, did you use a formula?
If your answer is yes, write the formula. If your answer is no, explain how you calculated.

1.1 1.2 **3.** Determine the area and perimeter of each figure.

a)

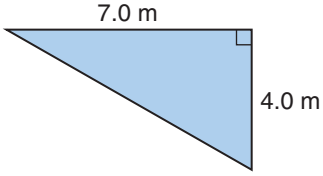


b)

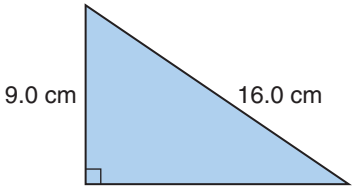


1.2 **4.** Determine each unknown length.

a)

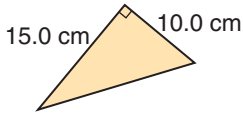


b)

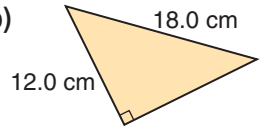


5. Determine the perimeter of each triangle.

a)

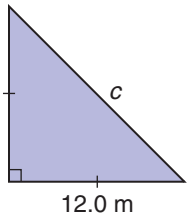


b)

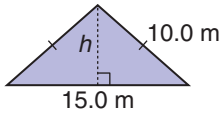


6. Determine each unknown length.

a)

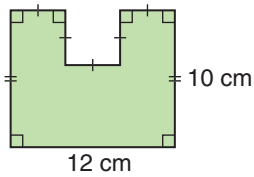


b)

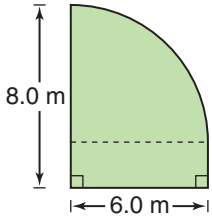


1.3 1.4 **7.** Determine the perimeter and area of each composite figure. How do you know your results are reasonable?

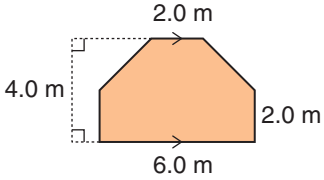
a) A rectangle with a square removed



b) A quarter circle on a rectangle



1.3 **8.** Andrew is painting 2 coats on 2 walls of his studio. Each wall looks like this:



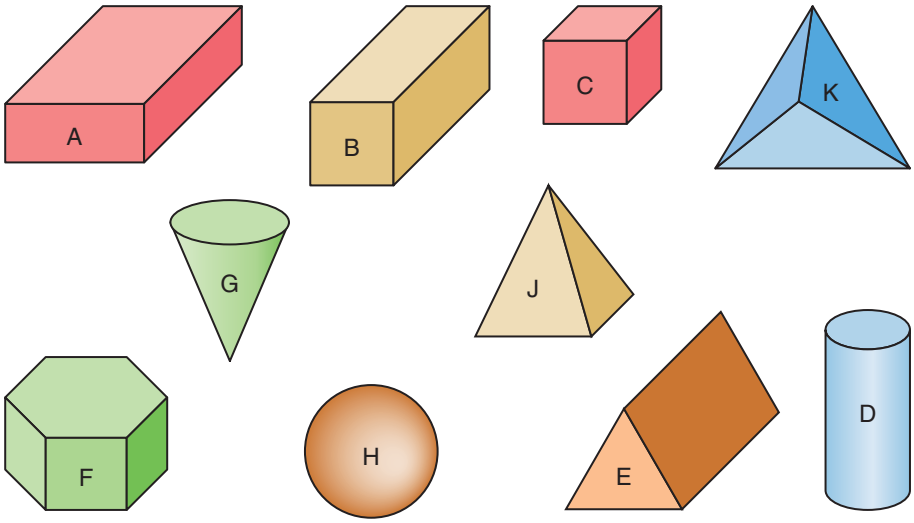
One can of paint covers 10.5 m^2 with one coat. How many cans of paint does Andrew need?
Justify your answer.

1.5 Volumes of a Prism and a Cylinder

Some buildings and bridges are supported by concrete columns called piers. A civil engineer calculates the volume of the piers so that enough concrete is ordered for a project.

Investigate Relating the Volumes of a Prism and a Cylinder

Which of these pictures represent prisms? Justify your answers. What would you need to know to determine the volume of each prism?

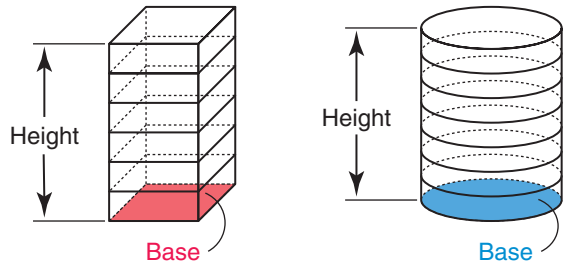


Reflect

- What is the same about all the prisms? What is different?
- Suppose each prism is filled with layers of congruent figures. Which figure would it be in each prism?
- Which object is a cylinder? Choose one of the prisms and compare it with the cylinder. What is the same about these two objects? What is different?

Connect the Ideas

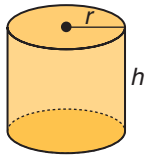
Compare a prism and a cylinder.
Each solid can be placed with its top directly above the base.
Visualize slicing the prism and the cylinder into layers.



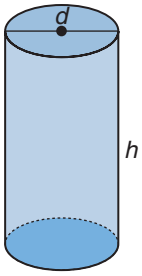
The area of each layer equals the area of the base.
The height of the layers is equal to the height of the prism and cylinder.
So, the volume of the solid equals the area of the base multiplied by the height.

The volume of a prism is:	The volume of a cylinder is:
V = base area × height	V = base area × height

The base area of a cylinder is: πr^2
The height is: h



So, the volume of a cylinder can also be written as:
$V = \pi r^2 h$



A hot-water tank is shaped like a cylinder with base diameter 56 cm and height 120 cm.
Its volume can be found using this formula: $V = \pi r^2 h$

The radius r is: $\frac{56 \text{ cm}}{2} = 28$ cm

Substitute: $r = 28$ and $h = 120$

$V = \pi \times 28^2 \times 120$
$\doteq 295\ 561.0$

The volume of the tank is about 300 000 cm^3.
1 cm^3 of volume = 1 mL of capacity
And, 1000 mL = 1 L

So, 300 000 cm^3 $= \frac{300\ 000}{1000}$ L
$= 300.000$ L

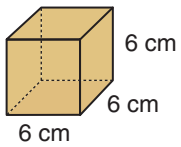
The capacity of the tank is about 300 L.

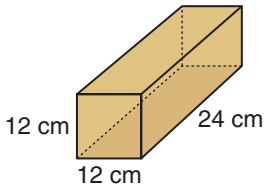
A label on the tank indicates that its capacity is 190 L.

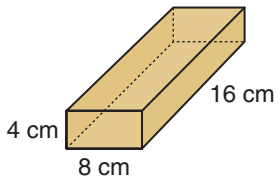
Why do you think the two capacities are so different?

Practice

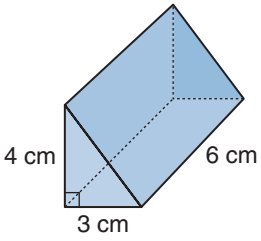
1. Determine the volume of each prism.

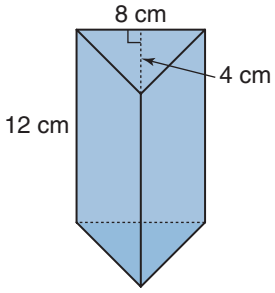
a)


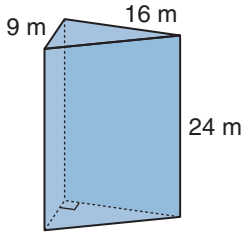
b)


c)


2. Determine the volume of each prism.

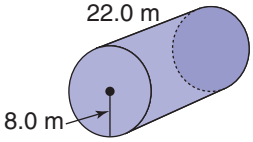
a)


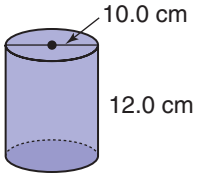
b)


c)


Need Help?
Read Connect the Ideas.

3. Determine the volume of each cylinder.

a)


b)


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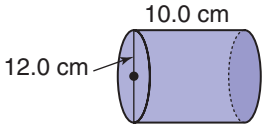
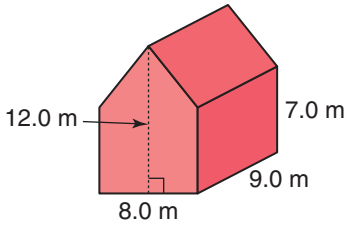


4. Pasta is sold in a box that is a rectangular prism.
The box that feeds 4 people measures 3 cm by 9 cm by 18 cm.
a) What is the volume of this box?
b) The company wants to produce a party-pack box of pasta.
Each dimension of the box will be doubled.
Will this be enough pasta for 16 people? Justify your answer.

5. **Assessment Focus** Hay bales come in different shapes and sizes.
Some are rectangular prisms. Others are cylindrical.
A rectangular bale is 75 cm by 20 cm by 14 cm.
A cylindrical bale has base diameter 150 cm and length 120 cm.
a) Sketch each bale. Which has the greater volume? Justify your answer.
b) About how many of the smaller bales have a total volume equal to that of the larger bale?

6. a) What is the volume of this barn?
b) Would this barn hold 1000 of the rectangular hay bales in question 5? How do you know?
c) Would this barn hold 1000 of the cylindrical hay bales in question 5? How do you know?
d) Are your results reasonable? Explain.

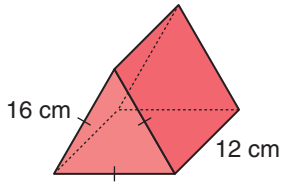


Sometimes we need to use the Pythagorean Theorem to calculate a length on a prism, before we find its volume.

Example

a) Determine the height of the base of this prism.

b) Determine the volume of this prism.



Solution

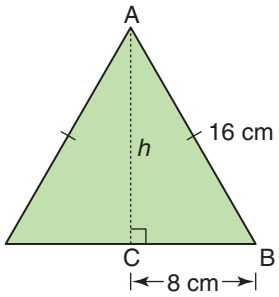
A base of a prism is not necessarily the bottom face.

a) Sketch the triangular base.

Let the height of the triangle be h.

The height bisects the base of the triangle.

Use the Pythagorean Theorem in $\triangle ABC$.



$$h^2 + 8^2 = 16^2$$
$$h^2 + 64 = 256$$
$$h^2 = 256 - 64$$
$$h^2 = 192$$
$$h = \sqrt{192}$$
$$h \doteq 13.86$$

The height of the base is about 14 cm.

b) The base area is: $\frac{1}{2} \times 16 \times 13.86 = 110.88$

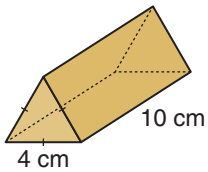
The length is: 12 cm

The volume is: base area $\times$ length $= 110.88 \times 12$
$= 1330.56$

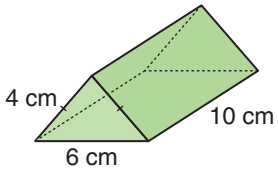
The volume of the prism is about 1331 cm^3.

7. A child's building block set has these triangular prisms. Determine the volume of wood in each block.

a) Equilateral triangular prism



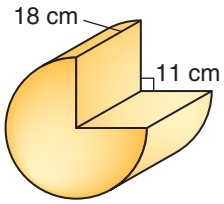
b) Isosceles triangular prism



8. **Take It Further** Many types of cheese are produced in cylindrical slabs. One-quarter of this slab has been sold.

a) What is the volume of this piece of cheese?

b) The mass of 1 cm^3 of cheese is about 1.2 g. What is the mass of the cheese shown here?



In Your Own Words

Why can you use the same formula to calculate the volumes of a prism and a cylinder? Include examples in your explanation.

1.6 Volume of a Pyramid

A company wants to know how much oil is contained in a new oilfield.
A geologist estimates the volume of oil underground. She models the layers of oil as cut-off pyramids and calculates their volumes.

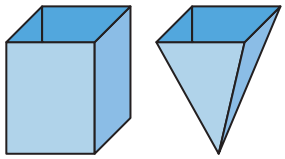
Investigate Relating the Volumes of a Prism and a Pyramid

A prism and pyramid with the same base and height are *related*.

Work with a partner.
You will need:

- a pyramid and a prism with congruent bases and equal heights
- sand or plastic rice

➢ Make a prediction. How do you think the volumes of this pyramid and prism compare?



Fill the pyramid with sand.

➢ Estimate how many pyramids of sand will fill the prism.

➢ Fill the prism to check your estimate.

➢ How is the volume of the pyramid related to the volume of the prism? How does this compare with your prediction?

Reflect

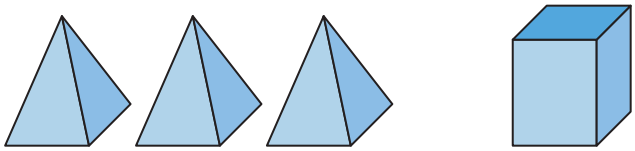
Compare results with classmates who used related prisms and pyramids with bases different from yours.

➢ Does the volume relationship depend on the shape of the base of the related pyramid and prism? Explain.

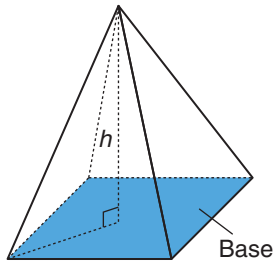
➢ Use what you know about the volume of a prism to write a formula for the volume of a related pyramid.

Connect the Ideas

The contents of three pyramids fit exactly into the prism.
These 3 volumes together… …are equal to this volume.

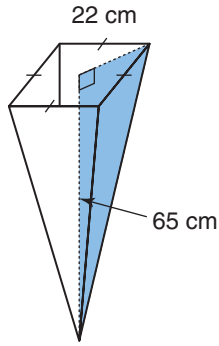


That is, the volumes of 3 pyramids are equal to the volume of the related prism.
So, the volume of a pyramid is one-third the volume of the related prism.



> $V = \frac{1}{3}Bh$, where B is the area of the base of the pyramid and h is the height of the pyramid.

To calculate how much plaster is needed to fill this mould, we calculate the volume of the pyramid.
The base of the pyramid is a square with side length 22 cm.



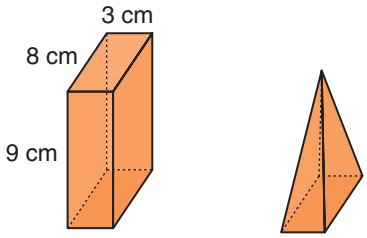
So, the base area is: $B = 22 \times 22 = 484$
The height of the pyramid is: $h = 65$
The volume of the pyramid is: $V = \frac{1}{3}Bh$

$$V = \frac{1}{3} \times 484 \times 65$$
$$\doteq 10\,486.67$$

The volume of plaster is about 10 487 cm^3.

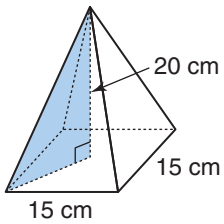
Practice

1. The prism and pyramid have the same base and height.
Determine each volume.

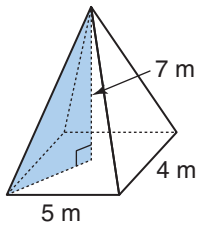


2. Determine the volume of each rectangular pyramid.

a)



b)



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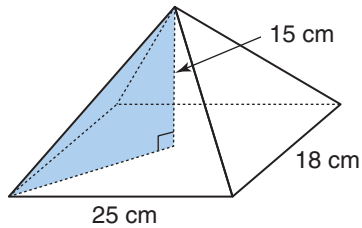
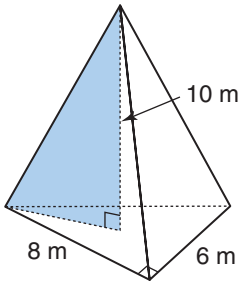


3. Determine the volume of each pyramid.

a)



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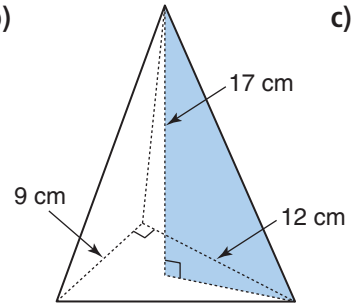

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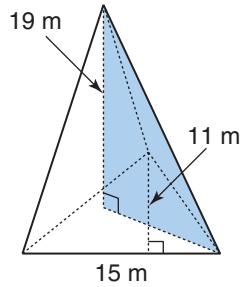
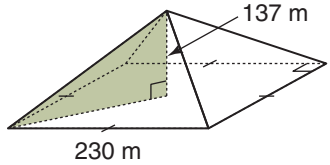


4. Pyramids have been constructed in many places around the world.
One of the most famous is the Great Pyramid of Giza.
It contains the burial chamber of Pharaoh Khufu.
Today, the pyramid is 137 m high.
When first constructed, it was 146.5 m high.

a) Sketch the original pyramid.
Label the sketch with the given measurements.

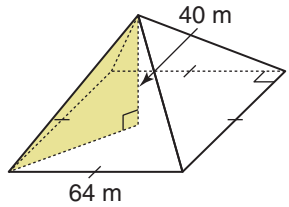
b) What volume of rock has been lost over the years?
Why do you think it has been lost?



5. A pyramid in Pune, India, can hold 5000 people.

a) What is the volume of air per person in the pyramid?

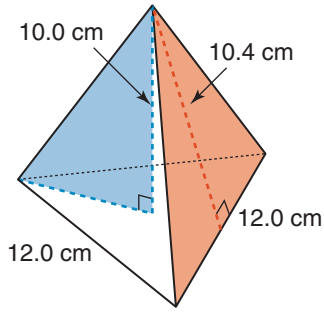
b) How do you know your answer is reasonable?



6. **Assessment Focus** A package for a frozen treat is a triangular pyramid. All edge lengths are 12.0 cm.
Each triangular face has height 10.4 cm.

a) Calculate the volume of the pyramid.
Show your work.

b) The package lists its contents as 200 mL.
Why are the contents in millilitres different from the volume in cubic centimetres?



$1\ cm^3 = 1\ mL$

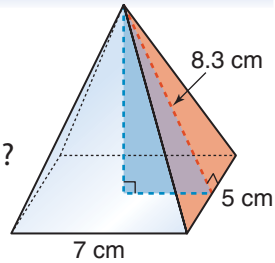
Sometimes the height of a pyramid is difficult to measure.
We can use the Pythagorean Theorem to calculate the height of the pyramid.

Example

A crystal paperweight is a pyramid with dimensions as shown.

a) What is the height of the pyramid?

b) What is the volume of crystal in the pyramid?

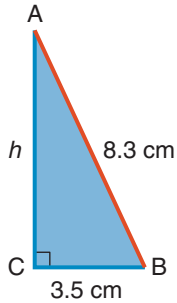


Solution

a) The height of the pyramid is one leg of a right triangle. The hypotenuse is 8.3 cm. The other leg is one-half the base of the other triangular face, or 3.5 cm.

Use the Pythagorean Theorem in △ABC.

$$h^2 + 3.5^2 = 8.3^2$$
$$h^2 + 12.25 = 68.89$$
$$h^2 = 68.89 - 12.25$$
$$h^2 = 56.64$$
$$h = \sqrt{56.64}$$
$$h \doteq 7.526$$



The height of the pyramid is about 7.5 cm.

b) The volume of the pyramid is: $V = \frac{1}{3}Bh$

The base area is: $B = 7 \times 5 = 35$

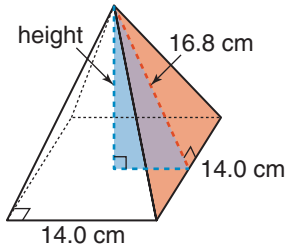
So, $V = \frac{1}{3} \times 35 \times 7.526 \doteq 87.80$

The volume of crystal in the pyramid is about 88 cm³.

7. A wooden pyramid has a square base with side length 14.0 cm. The height of each triangular face is 16.8 cm.

a) Determine the height of the pyramid.

b) Determine the volume of wood in the pyramid.



8. **Take It Further** The volume of the Great Pyramid of Cholula in Mexico is estimated to be 4.5 million cubic metres. It is 66 m high. What is the side length of its square base? Justify your answer.

In Your Own Words

How is the volume of a pyramid related to the volume of a prism? Include diagrams in your explanation.

1.7 Volume of a Cone

Large quantities of grains are sometimes stored in conical piles.
Often, only the height of the pile is known.
Techniques have been developed to estimate the radius of the pile.
The volume can then be calculated.

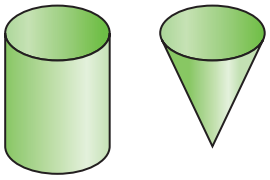
Investigate Relating the Volumes of a Cylinder and a Cone

A cylinder and cone with the same base and height are *related*.

Work with a partner.
You will need:

- a cylinder and a cone with congruent bases and equal heights
- sand or plastic rice

➢ Make a prediction. How do you think the volumes of this cylinder and cone compare?



Fill the cone with sand.

➢ Estimate how many cones of sand will fill the cylinder.

➢ Fill the cylinder to check your estimate.

➢ How is the volume of the cone related to the volume of the cylinder? How does this compare with your prediction?

Reflect

Compare results with classmates who used related cylinders and cones with bases different from yours.

➢ Does the volume relationship depend on the size of the base of the related cone and cylinder? Explain.

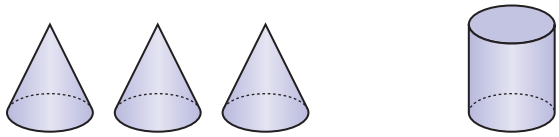
➢ Use what you know about the volume of a cylinder to write a formula for the volume of a related cone.

Connect the Ideas

A cone and a cylinder with the same base and height are related.

The relationship between the volumes of a cone and its related cylinder is the same as that for a pyramid and its related prism.

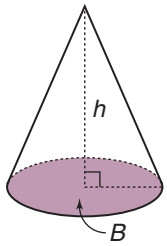
These 3 volumes together … … are equal to this volume.



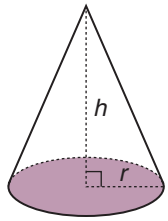
The volume of a cone is one-third the volume of its related cylinder.

So, the volume of a cone is: $V = \frac{1}{3}Bh$

where B is the area of the base of the cone and h is the height of the cone.

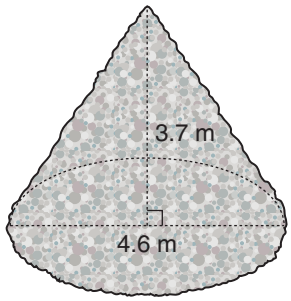


The volume of a cylinder is: $V = \pi r^2 h$
So, the volume of its related cone is:
$V = \frac{1}{3}\pi r^2 h$ or $\frac{\pi r^2 h}{3}$ or $\pi r^2 h \div 3$



The volume of a cone is $V = \frac{1}{3}\pi r^2 h$,
where r is the base radius and h is the height.

When a conveyor belt drops gravel, the gravel forms a cone.
This cone is 3.7 m high and has a base diameter of 4.6 m.

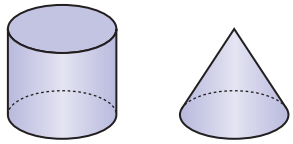


The volume of gravel is: $V = \frac{1}{3}\pi r^2 h$
$r = \frac{4.6\text{ m}}{2} = 2.3\text{ m}$
Substitute: $r = 2.3$ and $h = 3.7$
$V = \frac{1}{3} \times \pi \times 2.3^2 \times 3.7$
$V \doteq 20.497$
The volume of gravel is about 20.5 m^3.

Practice

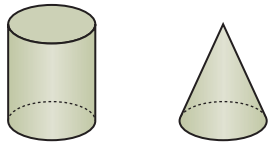
1. The cylinder and cone in each pair have the same base and height.
The volume of each cylinder is given. Determine the volume of each cone.

a)



$V = 42\ \text{cm}^3$ $V = ?$

b)

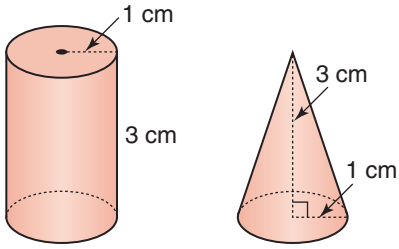


$V = 19.2\ \text{cm}^3$ $V = ?$

Need Help?
Read Connect the Ideas.

2. The cylinder and cone in each pair have the same base and height.
Determine each volume.

a)



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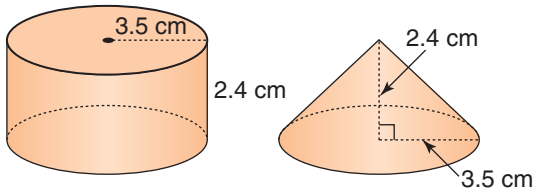
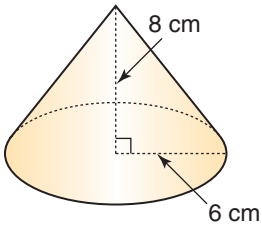


3. Determine the volume of each cone.

a)



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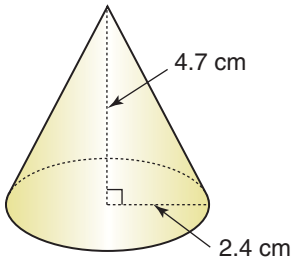
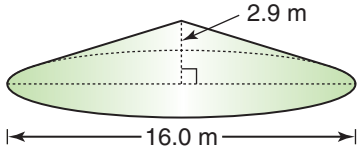
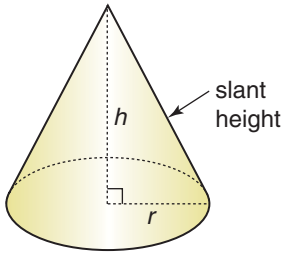


c)



4. An ice-cream cone has diameter 6 cm and height 12 cm.
What is the volume of the cone? Justify your answer.

Sometimes the height of a cone is not given.
We can measure the **slant height** and radius,
then use the Pythagorean Theorem to calculate the height.

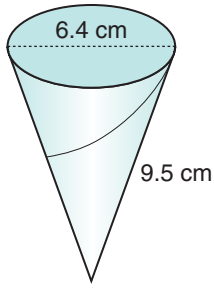


Example

A paper drinking cup is a cone.
The base has diameter 6.4 cm.
The slant height is 9.5 cm.

a) Determine the height of the cone.

b) Determine the volume of water that will fill the cup.

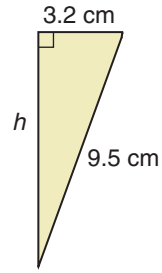


Solution

a) The radius of the cone is: $r = \frac{6.4 \text{ cm}}{2} = 3.2 \text{ cm}$

Use the Pythagorean Theorem:

$$\begin{aligned} h^2 + 3.2^2 &= 9.5^2 \\ h^2 + 10.24 &= 90.25 \\ h^2 &= 90.25 - 10.24 \\ h^2 &= 80.01 \\ h &= \sqrt{80.01} \\ h &\doteq 8.945 \end{aligned}$$



The cone is about 9 cm high.

b) The volume of water the cone will hold is equal to the volume of the cone.

Use: $V = \frac{1}{3}\pi r^2 h$

Substitute: $r = 3.2$ and $h = 8.945$

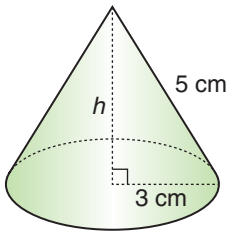
$V = \frac{1}{3} \times \pi \times 3.2^2 \times 8.945$

$V \doteq 95.92$

The volume of water is about 96 cm^3.

5. Determine the height and volume of each cone.

a)



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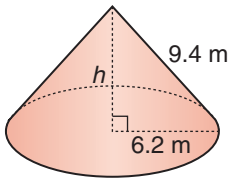

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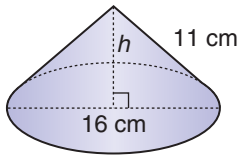


6. **Assessment Focus** A pile of sand is a cone.
The base diameter is 3.8 m and the slant height is 4.0 m.

a) What is the volume of sand in the pile? Include a diagram.

b) How do you know your answer is reasonable?

7. **Take It Further** A hill can be approximated as a cone.
Its circumference is about 3 km. Its slant height is about 800 m.
What is the approximate volume of soil in the hill?

Recall that
1 km = 1000 m.

In Your Own Words

How is the volume of a cone related to the volume of a cylinder?
Include diagrams in your explanation.

1.8 Volume of a Sphere

Some companies have experimented with making ball bearings in space. In space, a molten lump of steel floats. As the steel cools and hardens, it forms a perfect sphere. This is simpler than the manufacturing process on Earth. How could an engineer determine how much molten steel would be needed to produce a particular size of sphere in space?

Investigate The Volume of a Sphere

Recall that
1 mL = 1 cm^3.

Work with a partner.
You will need:

- a graduated cylinder
- a sphere that sinks
- water

➢ Estimate the volume of the sphere. Explain your estimation strategy.

➢ Record the volume of water in the graduated cylinder.
Place the sphere in the cylinder.
Record the new level of water.
Calculate the volume of the sphere in cubic centimetres.
How does the actual volume compare with your estimate?

➢ Find a way to measure the radius of your sphere.

➢ The volume of a sphere depends on the cube of its radius, r. For your sphere, calculate $r \times r \times r$, or r^3.

Reflect

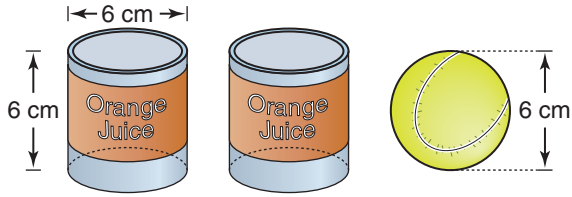
➢ Divide the volume, V, by r^3.

➢ Compare results with several other pairs of classmates. What do you notice about your values of $V \div r^3$?

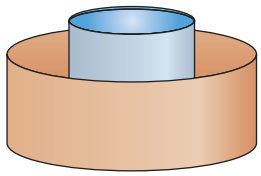
Connect the Ideas

An old tennis ball has the same diameter as a frozen juice can. This coincidence can be used to relate the volume of the ball to the volume of the can.

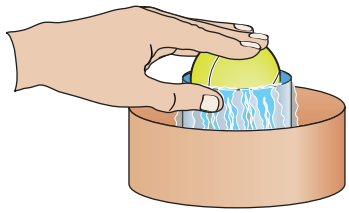
➢ Two matching empty cans are cut so their heights equal the diameter of the ball.



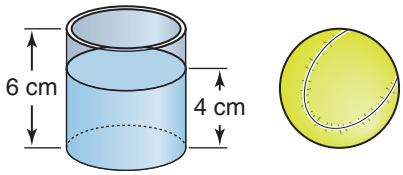
➢ One can is filled with water.



➢ The ball is soaked with water then pushed into the can right to the bottom. Water overflows into a tray.



➢ The volume of water in the tray equals the volume of the ball. This water is poured into the other can. The height of the water is measured.

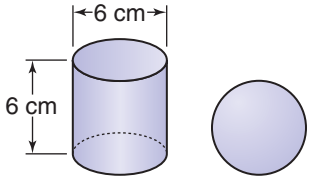


➢ The height of the water is 4 cm. The height of the can is 6 cm.

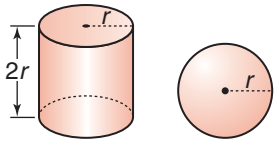
Since the cans are congruent, the ratio of the volumes of water is equal to the ratio of the heights.

Since 4 cm is $\frac{2}{3}$ of 6 cm, the volume of the ball is $\frac{2}{3}$ the volume of the can.

Volume of a sphere $= \frac{2}{3} \times$ Volume of a cylinder into which the sphere just fits



The height of this cylinder equals its diameter.
The volume of the cylinder is about 170 cm^3.
The volume of the sphere with the same diameter is about:
$\frac{2}{3} \times 170\ cm^3 \doteq 113\ cm^3$

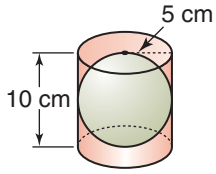


The volume of this cylinder is $\pi r^2 \times 2r = 2\pi r^3$
The volume of the sphere is:
$\frac{2}{3} \times 2\pi r^3 = \frac{4}{3}\pi r^3$

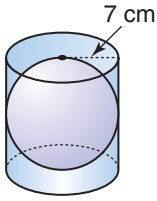
The volume V of a sphere with radius r is: $V = \frac{4}{3}\pi r^3$

Practice

1. The height of this cylinder is twice its radius.
Determine the volumes of the cylinder and the sphere.

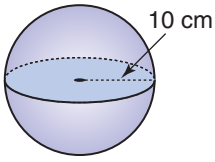


2. The height of this cylinder is twice its radius.
Determine the volume of the sphere.



3. Determine the volume of each sphere.

a)



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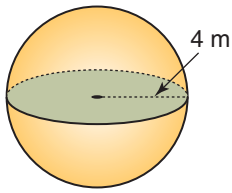

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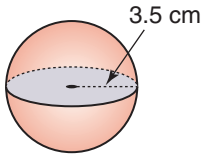
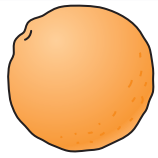


Many objects are approximately spherical.
Their volumes can be estimated using the formula for the volume of a sphere.

Example

An orange is approximately spherical.
Its diameter is 10 cm.
What is the volume of the orange?



Solution

To determine the volume of the orange, use: $V = \frac{4}{3}\pi r^3$

$r = \frac{10\text{ cm}}{2} = 5\text{ cm}$. Substitute: $r = 5$

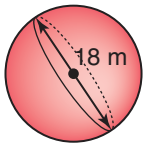
$V = \frac{4}{3} \times \pi \times 5^3$

$V \doteq 523.599$

The volume of the orange is about 524 cm^3.

4. Determine the volume of each sphere.

a)



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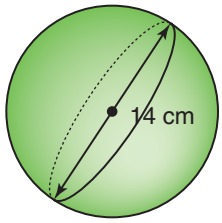

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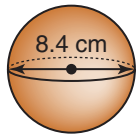
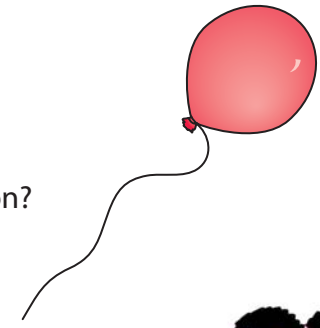




5. An inflated balloon approximates a sphere with radius 11.5 cm.
A student's lung capacity is 3.6 L.

a) How many breaths does the student use to inflate the balloon? What assumptions did you make?

b) How do you know your answer is reasonable?

6. **Assessment Focus** Lyn has a block of wood that measures 14 cm by 14 cm by 14 cm.
She is making a wooden ball in tech class.

a) What is the volume of wood in the block?

b) What is the largest possible diameter for the ball?

c) What is the volume of the wooden ball?

d) What volume of wood is cut off the block to make the ball? What assumptions did you make?

7. **Take It Further** Meighan is selling ice-cream cones at the fall fair.
Each carton of ice cream is 20 cm by 11 cm by 24 cm.
The ice-cream scoop makes a sphere of ice cream, with diameter 8 cm.

a) How many scoops should Meighan get from each carton?

b) Each carton of ice cream costs \$4.29. How much does each scoop cost?

c) Meighan pays \$1.99 for a package of 12 sugar cones. Suggest a price Meighan should charge for each single-scoop and double-scoop cone. Justify your answer.

In Your Own Words

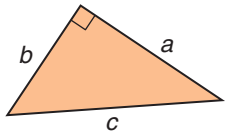
How is the volume of a sphere related to the volume of a cylinder? Include diagrams in your explanation.

Chapter Review

What Do I Need to Know?

Pythagorean Theorem

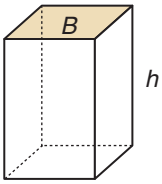
$a^2 + b^2 = c^2$, where a and b are the legs of a right triangle and c is the hypotenuse



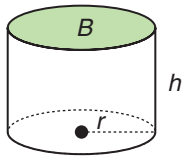
Volume Formulas

See the Expanded Glossary for the formulas for areas and perimeters of figures.

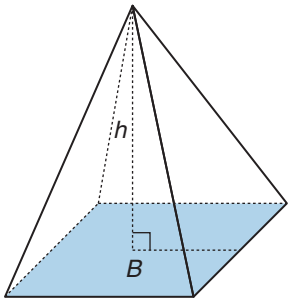
Volume of a prism is base area times height.
$V = Bh$



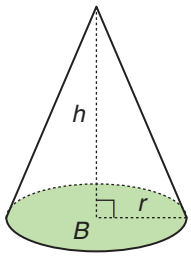
Volume of a cylinder is base area times height.
$V = \pi r^2 h$



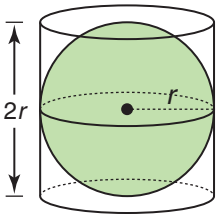
Volume of a pyramid is one-third the base area times the height.
$V = \frac{1}{3} Bh$



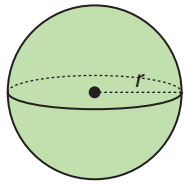
Volume of a cone is one-third the base area times the height.
$V = \frac{1}{3} \pi r^2 h$



Volume of a sphere is two-thirds the volume of a cylinder into which the sphere just fits.



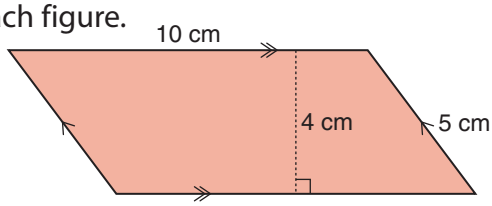
Volume of a sphere is four-thirds π times the cube of the radius.
$V = \frac{4}{3} \pi r^3$



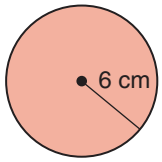
What Should I Be Able to Do?

1.1 **1.** Determine the perimeter and area of each figure.

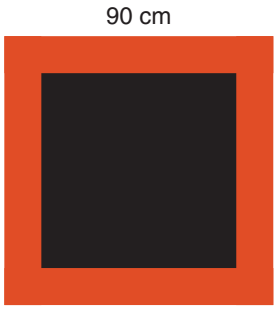
a)



b)



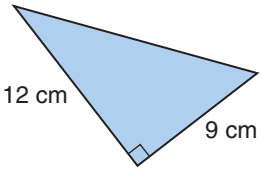
2. A hurricane warning flag is a square with side length 90 cm.
The red border is 12 cm wide.



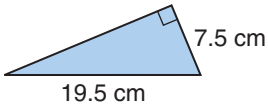
a) What is the area of black material?
b) What is the area of red material?

1.2 **3.** Determine each unknown length.

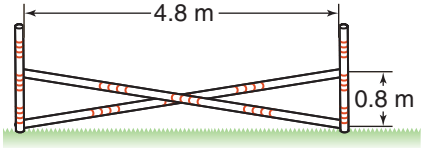
a)



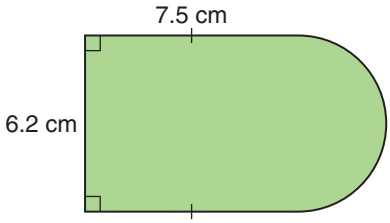
b)



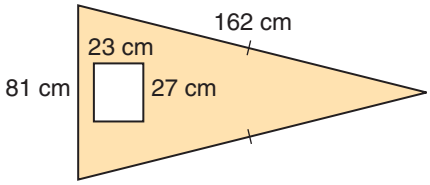
4. Jean set up cross poles for his horse to jump. How long is each cross pole?



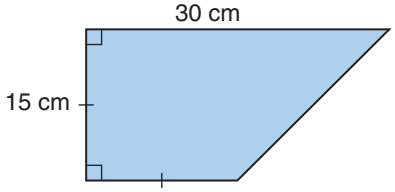
1.3 **5.** Determine the area of this figure. The curve is a semicircle.



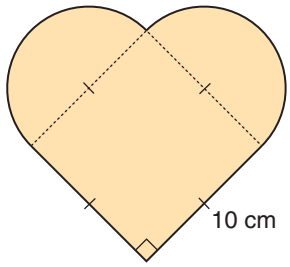
6. The sailing regatta committee has this flag to show a fourth place finish. What is the area of the red material in the flag?



1.4 **7.** Determine the perimeter of this figure.



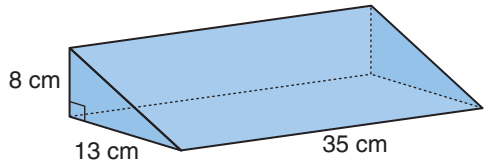
8. A large greeting card has the shape of a square, with a semicircle on each of two sides.



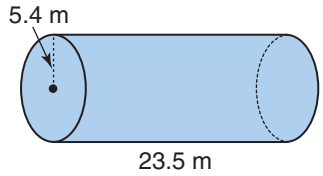
a) There is a ribbon around the perimeter of the card.
How long is this ribbon?
b) How do you know your answer is reasonable?

1.5 **9.** Determine the volume of each object.

a)



b)

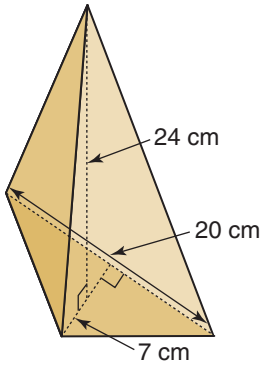


10. A tray of lasagna for 4 people is 19 cm wide, 24 cm long, and 7 cm deep.

a) Suppose the length and width of the tray are doubled. How many people should this new tray feed? Explain your answer. Include a diagram.

b) Suppose each dimension of the tray is doubled. How many people should the larger tray feed? Justify your answer.

1.6 **11.** Determine the volume of the pyramid.

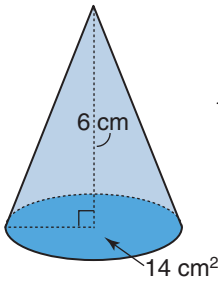


12. The Katimavik Pavilion at Expo '67 is a huge square pyramid. Its base is 20.0 m by 20.0 m. Its height is 14.1 m.

a) What is the volume of the pyramid?

b) How do you know your answer is reasonable?

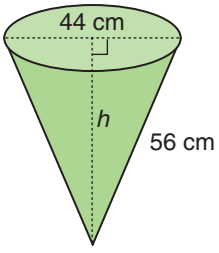
1.7 **13.** a) What is the volume of the cone?



b) What is the height of this related cylinder?

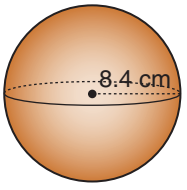
c) What would the cylinder's height have to be for it to have the same volume as the cone? Check your answer.

14. Sebastian is filling a conical piñata. How much space is there for candy?

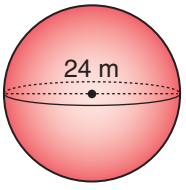


1.8 **15.** Determine the volume of each sphere.

a)



b)



16. In February 2003, Andy Martell of Toronto set a world record for the largest ball of plastic wrap. The ball was approximately spherical. Its diameter was about 43.6 cm.

a) What was its volume?

b) How do you know your answer is reasonable?

Practice Test

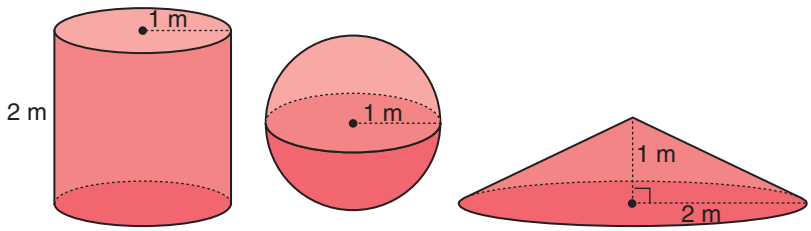
Multiple Choice: Choose the correct answer for questions 1 and 2.

1. A circle has diameter 16 cm. What is its approximate area?

A. 631.65 cm^2 **B.** 50.27 cm^2 **C.** 201.06 cm^2 **D.** 804.25 cm^2

2. Which statement is true for the volumes of these objects?

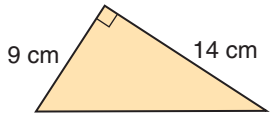
A. All volumes are equal.
B. The cylinder has the least volume.
C. Two volumes are equal.
D. The sphere has the greatest volume.



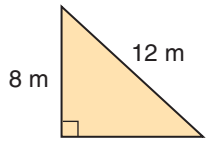
Show all your work for questions 3 to 6.

3. Knowledge and Understanding Determine each unknown length.

a)



b)

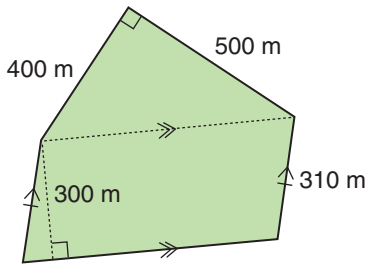


4. Application The town works department is ordering materials for a new park. It has the shape of a parallelogram with a right triangle on one side. Here is a plan of the park.

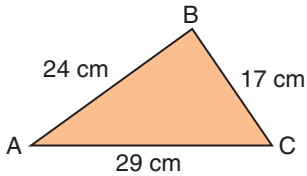
a) Which measure must you determine to find the area of the park? What is the park's area?

b) The park is to be fenced except for the gated entrances. There will be four 2.25-m gates in the fence. How much fencing needs to be ordered?

Justify your answers.



5. Communication In △ABC, is ∠ABC a right angle? Justify your answer.



6. Thinking Cameron is filling a drink cooler with bags of ice. How many bags of ice are needed to half fill the cooler? What assumptions did you make?